PROBLEMA RESUELTO 3

Calcule la integral impropia

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+r^2)^{3/2}}$$

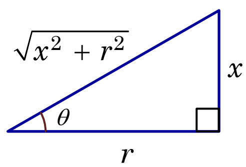
Donde r es un constante

Solución

Primero se debe calcular la integral indefinida

$$\int \frac{dx}{(x^2+r^2)^{3/2}}$$

Para calcular esta integral es necesario utilizar sustitución trigonométrica. Hacemos



$$\tan\theta = \frac{x}{r}$$

$$x = r\tan\theta$$

$$dx = r\sec^2\theta\, d\theta$$

$$\int \frac{dx}{(x^2+r^2)^{3/2}} = \int \frac{r\sec^2\theta d\theta}{\left[(r\tan\theta)^2+r^2\right]^{3/2}}$$

$$= \int \frac{r\sec^2\theta d\theta}{\left[r^2\tan^2\theta+r^2\right]^{3/2}}$$

$$= \int \frac{r\sec^2\theta d\theta}{\left[r^2(\tan^2\theta+1)\right]^{3/2}} = \int \frac{r\sec^2\theta d\theta}{\left[r^2(\sec^2\theta)\right]^{3/2}}$$

$$= \int \frac{r\sec^2\theta d\theta}{r^3\sec^3\theta} = \frac{1}{r^2}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{r^2}\int \cos\theta d\theta$$

$$= \frac{1}{r^2}\operatorname{sen}\theta + c$$

Como $\operatorname{sen}\theta = \frac{x}{\sqrt{x^2+r^2}}$ se tiene que

$$\int \frac{dx}{(x^2+r^2)^{3/2}} = \frac{1}{r^2}\frac{x}{\sqrt{x^2+r^2}} + c$$

Ahora se procede a evaluar la integral impropia

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + r^2)^{3/2}} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{(x^2 + r^2)^{3/2}} + \lim_{t\to\infty} \int_0^t \frac{dx}{(x^2 + r^2)^{3/2}}$$

$$= \lim_{t\to-\infty} \left[\frac{1}{r^2} \frac{x}{\sqrt{x^2 + r^2}}\right]_t^0 + \lim_{t\to\infty} \left[\frac{1}{r^2} \frac{x}{\sqrt{x^2 + r^2}}\right]_0^t$$

$$= \lim_{t\to-\infty} \left[\frac{1}{r^2} \frac{0}{\sqrt{0 + r^2}} - \frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right] + \lim_{t\to\infty} \left[\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}} - \frac{1}{r^2} \frac{0}{\sqrt{0 + r^2}}\right]$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right) + \lim_{t\to\infty} \left(\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right)$$

Calculando los límites

$$\lim_{t\to-\infty} \left(-\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right) = -\frac{1}{r^2} \lim_{t\to-\infty} \left(\frac{t}{\sqrt{t^2 + r^2}}\right) \frac{\frac{1}{t}}{\frac{1}{t}} = -\frac{1}{r^2} \lim_{t\to-\infty} - \frac{1}{r^2} \lim_{t\to-\infty} \left(\frac{1}{\frac{\sqrt{t^2 + r^2}}{t}}\right)$$

$$= -\frac{1}{r^2} \lim_{t\to-\infty} \left(\frac{1}{-\sqrt{1 + \frac{r^2}{t^2}}}\right) = -\frac{1}{r^2} \left(\frac{1}{-\sqrt{1 + 0}}\right) = \frac{1}{r^2}$$

$$\lim_{t\to\infty} \left(\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right) = \frac{1}{r^2} \lim_{t\to\infty} \left(\frac{t}{\sqrt{t^2 + r^2}}\right) \frac{\frac{1}{t}}{\frac{1}{t}} = \frac{1}{r^2} \lim_{t\to\infty} \left(\frac{1}{\frac{\sqrt{t^2 + r^2}}{t}}\right)$$

$$= \frac{1}{r^2} \lim_{t\to\infty} \left(\frac{1}{\sqrt{1 + \frac{r^2}{t^2}}}\right) = \frac{1}{r^2} \left(\frac{1}{\sqrt{1 + 0}}\right) = \frac{1}{r^2}$$

Ahora ya se puede obtener el resultado de la integral impropia

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + r^2)^{3/2}} = \lim_{t\to-\infty} \left(-\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right) + \lim_{t\to\infty} \left(\frac{1}{r^2} \frac{t}{\sqrt{t^2 + r^2}}\right)$$

$$= \frac{1}{r^2} + \frac{1}{r^2}$$

$$= \frac{2}{r^2}$$